Grade 9/10 Math Circles
March 20, 2024
Probability I

Probability is the area of mathematics which studies how we can handle randomness and uncertainty. Everyday examples include flipping a coin or drawing names from a hat. Today's session will focus on building up a framework which allows us to formalize probability mathematically.

Motivating Problem

Our goal today is to be able to be able to answer the birthday problem: In a group of people, what is the probability that at least 2 have the same birthday?

Sets

Probability relies heavily on set theory, so we begin with an overview of sets.

Definition 1. A *set* is an unordered collection of objects, or elements.

The notation $x \in A$ means that x is an element of A, and $x \notin A$ means that x is not in A.

We can define a set by describing everything in it or by listing elements inside of curly brackets. Sets can contain any sort of object - not just numbers.

Example 1. The following are all sets:

- $\{1, 2, 3\}$
- {dogs, cats, apples, trees}
- All math circles students.

Note that the order in which elements are written and the number of times they appear does not matter. So, the sets $\{1, 2, 3\}$, $\{3, 2, 1\}$, and $\{1, 1, 2, 3\}$ are all the same.

It is often useful to discuss the set containing nothing, so it has a special name.

Definition 2. The *empty set* contains no elements. It is written as $\{\}$ or $\emptyset$.

A subset is a set made up of elements from another set.

Definition 3. A is a subset of B, written $A \subseteq B$, if every element in A is also in B.

A subset could contain any number of elements from the original set. In fact, for any set A, we can always say that $A \subseteq A$ and $\emptyset \subseteq A$.

Example 2. The following are all subsets of $S = \{1, 2, 3\}$:

- $\emptyset$
- $\{1\}$
- $\{1, 2\}$
- $\{1, 2, 3\}$

But, $\{1, 5\}$ is **not** because $5 \notin A$.

Exercise 1. Is {AB} a subset of {A,B,C}?

Exercise 2. List all subsets of $S = \{x, y, z\}$.

Probability Spaces

Now that we have this language from set theory, we can use it to set up a framework in which we can discuss the probabilities of various outcomes.

Definition 4. An *experiment* is any activity that produces one result, called an *outcome*, out of a set of possible results, called the *sample space*.

Example 3. Experiment: flipping a coin
Sample space: {heads, tails}
Outcome: heads

We often want to consider more than one outcome at a time, especially if a set of outcomes is particularly meaningful.

Definition 5. An *event* is a subset of the *sample space*.

Example 4. Experiment: rolling a six-sided die
Sample space: $\{1, 2, 3, 4, 5\}$
Events: even roll - $\{2, 4, 6\}$, roll less than 3 - $\{1, 2\}$

Example 5. Experiment: taking an exam
Sample space: letter grades {A, B, C, D, F}
Events: passing grade - {A, B, C, D}, getting an A - {A}

Exercise 3. Give an example of an experiment, its sample space, and two events.

Multiple Events

Sometimes, we might want to consider the relationship between multiple events. The following definitions from set theory will help us do so.

Definition 6. The *intersection* of two sets A and B, written $A \cap B$, is the set of all elements in **both** A and B.

Example 6. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 6\}$. Then,

- $A \cap B = \{1, 3\}$
- $A \cap C = \{2\}$
- $B \cap C = \emptyset$

It is often important to know if two sets share any elements.

Definition 7. Two sets are *disjoint* if their intersection is the empty set.

In the above example, the sets B and C are disjoint.

Definition 8. The *union* of two sets A and B, written $A \cup B$, is the set of all elements in **at least one of** A or B.

Example 7. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 6\}$. Then,

- $A \cup B = \{1, 2, 3, 5\}$
- $B \cup C = \{1, 2, 3, 4, 5, 6\}$

Exercise 4. Suppose $A \subseteq B$. What are $A \cap B$ and $A \cup B$?

Exercise 5. Suppose you roll a six-sided die. Consider the events $O =$ odd roll, $E =$ even roll, $P =$ prime roll, $L =$ less than 3. Find:

1. $E \cap P$
2. $O \cap L$
3. $E \cup L$
4. $P \cup O$

Bonus: Are any of the four events disjoint?

Probability Measures

Now that we have the setup of an experiment, we can think about how to quantify the likelihood of various events. Let's first consider situations in which every outcome in our sample space has the same probability, like flipping a coin or rolling a die.

An intuitive way to find the probability of an event is to compare the number of outcomes in it to the total number of outcomes. So, we need to define the size of a set.

Definition 9. The *size* of a set A, written $n(A)$, is the number of elements in it.

Example 8.

- $n(\emptyset) = 0$
- $n(\{\text{x, y, z}\}) = 3$
- If θ is the set of all letters in the alphabet, then $n(\theta) = 26$.

We can now define our first probability function!

Let S be a sample space with finitely many outcomes, all of which are equally likely. The probability of an event $E \subseteq S$ is

$$P(E) = \frac{n(E)}{n(S)}.$$

That is, the likelihood of the experiment producing an outcome in E is equal to the number of outcomes in E divided by the total number of possible outcomes.

Example 9. Suppose you choose a random card from a standard deck of 52 cards. Find the probability of the following events:

1. $A =$ drawing an ace
2. $F =$ drawing a face card
3. $H =$ drawing a heart

Solution: Note that each card is equally likely to be drawn.

1. There are 4 aces in a deck, so $P(A) = \dfrac{n(\text{ace})}{n(\text{any card})} = \dfrac{4}{52}$.
2. There are 12 face cards (3 per suit), so $P(F) = \dfrac{n(\text{faces})}{n(\text{any card})} = \dfrac{12}{52}$.
3. There are 13 cards of each suit, so $P(H) = \dfrac{n(\text{heart})}{n(\text{any card})} = \dfrac{13}{52}$.

We can also consider the probability of combinations of events.

Example 10. Using the same setup as the previous example, find:

1. $P(A \cap H)$
2. $P(A \cup F)$

Solution:

- There is only 1 ace of hearts, so $P(A \cap H) = \dfrac{n(\text{ace of hearts})}{n(\text{any card})} = \dfrac{1}{52}$.
- There are 4 aces and 12 face cards. There is no overlap, so
  $P(A \cup F) = \dfrac{n(\text{aces}) + n(\text{faces})}{n(\text{any card})} = \dfrac{4+12}{52} = \dfrac{16}{52}$.

Exercise 6. Suppose you roll a six-sided die. Find the probabilities of the following events:

1. A = Rolling a 2
2. B = Rolling an odd number
3. $A \cap B$
4. $A \cup B$

One reason that this probability measure works so well is that it has some very nice properties. Specifically:

1. The probability of an event is between 0 and 1, which we can think of as 0% and 100%.
2. The probability of the experiment producing some outcome from the sample space is 100%.
3. If two events don't share any outcomes, then the probability of their union is equal to the sum of their probabilities.

Sometimes, not every outcome has the same probability. We can generalize the idea of our probability function by requiring it to have the above properties.

Definition 10. Let S be a sample space containing finitely many outcomes. A *probability measure* on S is a function P which maps events from S to values. For any events $E, F \subseteq S$, the following properties must hold:

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P(E \cup F) = P(E) + P(F)$ whenever $E \cap F = \emptyset$

Example 11. Suppose you roll a weighted six-sided die with a 25% chance of rolling a 1 and a 15% chance of rolling any other number. Then,

- $P(\{1\}) = 0.25$
- $P(\{2\}) = 0.15$
- $P(\{1\} \cup \{2\}) = P(\{1\}) + P(\{2\}) = 0.4$

Exercise 7. Suppose that a spinner has red, green, and blue spaces. If $\mathrm{P(red)} = 0.2$ and $\mathrm{P(green)} = 0.3$, what is $\mathrm{P(blue)}$?

Probability Rules

The rules for a probability measure only tell us how to calculate probabilities of unions when events are disjoint. The following rule tells us how to do this even when the events share some outcomes.

Union Rule: For any probability measure P and events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notice that when A and B are disjoint, their intersection is the empty set, and we get that $P(A \cup B) = P(A) + P(B)$. This agrees with what we required in the definition of our probability measure.

Example 12. Suppose you are rolling a six-sided die. Find that probability that you roll either an even number or a number greater than 4.

Solution:

$$\begin{aligned} P(\{2,4,6\} \cup \{5,6\}) &= P(\{2,4,6\}) + P(\{5,6\}) - P(\{2,4,6\} \cap \{5,6\}) \\ &= P(\{2,4,6\}) + P(\{5,6\}) - P(\{6\}) \\ &= 3/6 + 2/6 - 1/6 = 4/6 \end{aligned}$$

Notice that we also could have calculated the above by directly taking the union. Let's check that doing this would give us the same answer:

$$P(\{2,4,6\} \cup \{5,6\}) = P(\{2,4,5,6\}) = 4/6.$$

Sometimes, it is much easier to calculate the probability that an event does not happen. The following definition allows us to make use of this.

Definition 11. The *complement* of an event A, written A^C, is the set of all outcomes from the sample space **not** in A.

Example 13. If the sample space is $\{1,2,3,4\}$ and $A = \{1,2\}$, then $A^C = \{3,4\}$.

We can now state the following rule.

Complement Rule: For any probability measure P and event A,

$$P\left(A^C\right) = 1 - P(A).$$

Example 14. Show that $P(\emptyset) = 0$.

Solution: The complement of the empty set is the entire sample space. So,

$$P(\emptyset) = 1 - P(\emptyset^C) = 1 - P(S) = 1 - 1 = 0.$$

We can also use this rule to solve the birthday problem! To see how we can do this, we start with a simpler version.

Example 15. Suppose that 3 friends were born in the same week. What is the probability that at least 2 of them were born on the same day?

Solution: Let E be the event that at least 2 friends are born on the same day. It is very difficult to count all the ways this could occur. We would have to consider all 7 days, the chance that all 3 share a birthday, etc. It is much easier to think about E^C, the ways that no one shares a birthday.

First, we consider the sample space S - all possible ways for the 3 birthdays to occur. There are 7 possibilities for each of the 3 friends, so $n(S) = 7^3$.

Next, we think about what needs to happen for each person to have a different birthday. The first person could be born on any of the 7 days. The second person cannot be born on the same day as the first, so there are only 6 days left. Similarly, the third person cannot be born on the same day as either of the first two, so there are 5 days left. We conclude that $n(E^C) = 7 \cdot 6 \cdot 5$.

Since people are equally likely to be born on any given day, we find that

$$P(E) = 1 - P\left(E^C\right) = 1 - \frac{n(E^C)}{n(S)} = 1 - \frac{7 \cdot 6 \cdot 5}{7^3} \approx 0.39$$

Example 16. Suppose that there are 23 people, none of whom were born on February 29. What is the probability that at least 2 people share a birthday?

Solution: Let E be the event that at least 2 people have the same birthday. Like the previous example, it is difficult to write out all the ways E could occur. So, we consider the probability of E^C, the chance nobody shares a birthday.

First, lets consider our sample space S - all possible ways for the 23 birthdays to occur. Since there are 365 days in a year, and 23 people, $n(S) = 365^{23}$.

Now, we can think about what has to happen for each person to have a different birthday. The first person could be born on any day, so there are 365 possibilities. The second person could be born on any other day, so there are 364 remaining possibilities. Continuing with this logic, the n-th person has $365 - (n - 1)$ possible birthdays. This gives us a total of $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365 - 22)$ ways for E^C to occur.

Since people are equally likely to be born on any day, we calculate that

$$P(E) = 1 - P(E^C) = 1 - \frac{n(E^C)}{n(S)} = 1 - \frac{365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365 - 22)}{365^{23}} \approx 0.51$$

So, when there are at least 23 people, there is over a 50% chance that two people have the same birthday!

Additional Exercises

1. Suppose that you roll two six-sided dice. What is the probability that you roll doubles? What is the probability that their sum is less than 11?

2. Suppose you draw a random card from a standard 52 card deck. What is the probability that you draw either an ace or a spade? What about the probability that you draw a face card or a spade?

3. Suppose you have a bag with identically wrapped chocolates. There are 3 white chocolates, 6 milk chocolates, and 1 dark chocolate. Your friend adds k dark chocolates to the bag and tells you that the probability of picking a white or dark chocolate is now 25%. What is k?

4. Suppose there are 10 red balls and 10 blue balls in a box. 9 of those balls have the number 1 written on them, while the remainder have number 2 written on them. Moreover, the probability of selecting a ball that is red or has a 1 on it is 13/20. Determine the probability of drawing a ball of each type (red and 1, red and 2, blue and 1, blue and 2).